The Mathematics of Juggling

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Introduction

This paper examines the mathematics behind juggling by looking at sequences of numbers that can define a juggling pattern.

In a sequence, odd positions refer to throws by the right and and even positions refer to throws by the left. The numbers in a sequence refer to when the object will be thrown again. For example, a 5 means the object will be thrown again 5 throws in the future. Based on this fact the higher the number the more time you need to accomplish the given number of throws before the object will be thrown again. Therefore the larger the number the higher the throw must be. These two facts show us that odd numbers refer to throws from one hand to the other and even numbers are throws from one hand to that same hand. Where;

0 - is an empty hand

1 - is a straight toss from one hand to the other

2 - is a short toss in one hand but in practicallity is just a held ball.

3 - is the normal cascade toss.

4 - is the normal one handed toss.

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$2x$ - is a throw from one hand to itself.

$2x + 1$ - is a throw from one hand to the other.

where $x \in \mathbb{N}$.

In general these patterns can be written as the sequence $a_0 a_1 \ldots a_{n-1}$ where each a_i is a number in the sequence and the sequence can be repeated when juggling. One way to examine these patterns is to look at the paths of the individual objects. To do this;

1. First write out the pattern repeated several times.
2. Start at a_0.
3. Draw a curved line from a_j to a_k where a_k is the value of a_j positions to the right of a_j.
4. Repeat step 3 at a_k.
5. Continue to repeat step 3 until you run out of a_k to map to.
6. Repeat steps 3-5 starting with the first a_i that does not have a line drawn to or from it.
7. You are finished when all a_i have a line drawn to or from them or if two lines get drawn to the same place, we call this a collision and the sequence can not be juggled.

Some examples of patterns:

The three ball cascade - 3 = 3 3 3 3 3 3 3 3 3 3 3 3 3

The three ball shower - 51 = 5 1 5 1 5 1 5 1 5 1 5 1 5 1

123 = 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3

441 = 4 4 1 4 4 1 4 4 1 4 4 1 4 4 1

342 = 3 4 2 3 4 2 3 4 2 3 4 2 3 4 2

132 = 1 3 2 1 3 2 1 3 2 1 3 2 1 3 2

5312 = 5 3 1 2 5 3 1 2 5 3 1 2 5 3 1 2

5 = 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5

81411 = 8 1 4 1 1 8 1 4 1 1 8 1 4 1 1 8 1 4 1 1

661515 = 6 6 1 5 1 5 6 6 1 5 1 5 6 6 1 5 1 5 6 6 1 5 1 5

Other legit patterns: 531, 552, 5551, 55550, 612, 4413, 52512, 7162, 5141, 5313, 561

Now we will examine the mathematics behind these patterns. This will include how to determine if a pattern can be juggled and how many objects are used in that pattern.

Let $A_n = \{(a_0, 0), (a_1, 1), \ldots, (a_{n-1}, n-1)\}$ be the set of ordered pairs defining a juggling sequence of length n. Where the first entry of the ordered pair is the number from the sequence and the second entry is the position of this number in the sequence. Let $P_n = \{0, 1, \ldots, n-1\}$ be the set of numbers representing the positions in the sequence. Then define the function $f : A_n \to P_n$ where $f(a_i, i) = j$ and $j \equiv (a_i + i) \mod n$.

Definition - A hand is missed (**missed hand**) if there exists an $x \in P_n$ such that for all $(a_i, i) \in A_n$, $f(a_i, i) \neq x$. Notice this is the definition of a non-surjective function.

Definition - A **collision** occurs if for any two $(a_i, i), (a_j, j) \in A_n$, with $(a_i, i) \neq (a_j, j)$ and $f(a_i, i) = f(a_j, j)$. Notice this is the definition of a non-injective function.

Definition - Let us call a juggling sequence **legit** if and only if there are no missed hands or collisions.

The Legit Theorem - A sequence is legit if and only if $f : A_n \to P_n$ is bijective.

First assume that the sequence is legit.

The sequence is legit there are no collisions or missed hands. Since there are no collisions for every $f(a_i, i) = f(a_j, j)$, then $(a_i, i) = (a_j, j)$, therefore f is injective. Also there are no missed hands, then for every $x \in P_n$ there is an $(a_i, i) \in A_n$ such that $f(a_i, i) = x$. Therefore f is surjective, so f is bijective.

Now assume that f is bijective.

Since f is injective, we have that if $f(a_i, i) = f(a_j, j)$, then $(a_i, i) = (a_j, j)$, therefore there are no collisions. For all $x \in P_n$, there is an $(a_i, i) \in A_n$ such that $f(a_i, i) = x$ since f is surjective, therefore there are no missed hands so the sequence is legit.

The Test

To test the legitimacy of a sequence, add each element of the sequence to its position in the sequence mod n. If each result is distinct, then the sequence is legit. (Notice that these steps follow from g being bijective).

Single Number Sequence Corollary - If a juggling sequence can be written as a single number, then it is legit.

Let $A_1 = \{(a_0, 0)\}$ and $P_1 = \{0\}$, then there is only one possible function from $f : A_1 \to P_1$ and that is $f(a_0) = 0$. This function is bijective, therefore the sequence is legit.

Remark - It is well known that for a function between two sets of the same finite order, the following are equivalent:

1. the function is injective,
2. the function is surjective,
3. the function is bijective.

The Missed Hand Theorem - If $f : A_n \to P_n$, then there is a collision if and only if there is a missed hand in the sequence.

Assume there is a collision in the sequence.

By the legit theorem, the sequence is not bijective. Since A_n and P_n have the same finite order, n, from the remark above, we have that the function is not surjective.

Therefore, by definition, there is a missed hand.

Assume there is a missed hand.

By the legit theorem, the sequence is not bijective. Since A_n and P_n have the same finite order, n, from the remark above, we have that the function is not injective.

Therefore, by definition, there is a collision.

Now define the function $g : A_n \to A_n$, by $g(a_i, i) = (a_j, j)$ with $j \equiv (a_i + i) \mod n$.

The Ball Theorem - If the sequence is legit, then the number of objects needed for the sequence, is equal to the average of the numbers that make up the sequence.

Define a cycle to be the sequence repeated a sufficient number of times for every object to be in its original position. Now let k be the number of sequences in one cycle and let b be the number of objects used to juggle the sequence. Recall that n is the length of the sequence. Notice that g is bijective by the nature of its construction and its relation to f. Since there are only a finite number of functions from a finite set to itself, the composition of g with itself must repeat. Therefore, k is finite. Now if we look at a cycle of length, kn, each object must travel through this entire cycle, since the sequence is legit. So the sum of the numbers for each object must be kn, in order for them to have travelled through one cycle. So the total sum of all of the numbers in the cycle add up to the number of objects times the sum of the numbers for each ball which is bkn. Then the average of the numbers, the sum of all numbers in the cycle divided by the total number of positions, is $\dfrac{bkn}{kn} = b$. Also the average of one cycle is the same as the average for the sequence, since the ks cancel, so the number of balls is equal to the sum of the numbers in the sequence divided by n.